# The Axioms of the Whitehead Russell Calculus

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## Abstract

This module notates the original axioms of the Whitehead-Russell calculus, the so called 'primitive propositions'. These five primitive propositions could be deduced by our four axioms.

## Specification

This document has the following specification:

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| Name: | proptheo2 |
| Version: | 1.00.00 |
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## References

This document uses the results of the following documents:

| | |
|---|---|
| Name: | propaxiom |
| Version: | 1.00.00 |
| Rule version: | 1.00.00 |
| Orgin: | propaxiom_1.00.00_1.00.00.qedeq |
| pdf: | propaxiom_1.00.00_1.00.00.pdf |

## Content

At first we show a little proposition to demonstrate the basic proof methods of propositional calculus:

**Theorem 0.1 (lem1).**

$$P \to (Q \vee P)$$

*Proof.*

| | | |
|---|---|---|
| 1 | $(P \to Q) \to ((A \vee P) \to (A \vee Q))$ | add axiom axiom4 |
| 2 | $(B \to Q) \to ((A \vee B) \to (A \vee Q))$ | replace $P$ by $B$ in 1 |
| 3 | $(B \to (Q \vee P)) \to ((A \vee B) \to (A \vee (Q \vee P)))$ | replace $Q$ by $Q \vee P$ in 2 |
| 4 | $((P \vee Q) \to (Q \vee P)) \to ((A \vee (P \vee Q)) \to (A \vee (Q \vee P)))$ | replace $B$ by $P \vee Q$ in 3 |
| 5 | $((P \vee Q) \to (Q \vee P)) \to ((\neg P \vee (P \vee Q)) \to (\neg P \vee (Q \vee P)))$ | replace $A$ by $\neg P$ in 4 |
| 6 | $((P \vee Q) \to (Q \vee P)) \to ((P \to (P \vee Q)) \to (\neg P \vee (Q \vee P)))$ | reverse abbreviation impl in 5 at occurence 1 |
| 7 | $((P \vee Q) \to (Q \vee P)) \to ((P \to (P \vee Q)) \to (P \to (Q \vee P)))$ | reverse abbreviation impl in 6 at occurence 1 |
| 8 | $(P \vee Q) \to (Q \vee P)$ | add axiom axiom3 |
| 9 | $(P \to (P \vee Q)) \to (P \to (Q \vee P))$ | MP with 8, 7 |
| 10 | $P \to (P \vee Q)$ | add axiom axiom2 |
| 11 | $P \to (Q \vee P)$ | MP with 10, 9 |

□

This is the first primitive proposition, its equal to our first axiom:

**Theorem 0.2 (prin1).**

$$(P \vee P) \to P$$

*Proof.*

| | | |
|---|---|---|
| 1 | $(P \vee P) \to P$ | add axiom axiom1 |

□

Now comes the second primitive proposition. It looks simular to our second axiom, but we have to use our first proposition to prove it:

**Theorem 0.3 (prin2).**

$$Q \to (P \vee Q)$$

*Proof.*

| | | |
|---|---|---|
| 1 | $P \to (Q \vee P)$ | add sentence lem1 |
| 2 | $P \to (A \vee P)$ | replace $Q$ by $A$ in 1 |
| 3 | $Q \to (A \vee Q)$ | replace $P$ by $Q$ in 2 |
| 4 | $Q \to (P \vee Q)$ | replace $A$ by $P$ in 3 |

□

The third primitive proposition:

**Theorem 0.4 (prin3).**

$$(P \vee Q) \to (Q \vee P)$$

*Proof.*

| | | |
|---|---|---|
| 1 | $(P \vee Q) \rightarrow (Q \vee P)$ | add axiom axiom3 |

□

The fourth primitive proposition was proved with the other primitve propositions by P. Bernays. Here comes the sledgehammer:

**Theorem 0.5 (prin4).**

$$(P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A))$$

*Proof.*

| | | |
|---|---|---|
| 1 | $(P \rightarrow Q) \rightarrow ((A \vee P) \rightarrow (A \vee Q))$ | add axiom axiom4 |
| 2 | $(P_7 \rightarrow Q) \rightarrow ((A \vee P_7) \rightarrow (A \vee Q))$ | replace $P$ by $P_7$ in 1 |
| 3 | $(P_7 \rightarrow P_8) \rightarrow ((A \vee P_7) \rightarrow (A \vee P_8))$ | replace $Q$ by $P_8$ in 2 |
| 4 | $(P_7 \rightarrow P_8) \rightarrow ((P_9 \vee P_7) \rightarrow (P_9 \vee P_8))$ | replace $A$ by $P_9$ in 3 |
| 5 | $P \rightarrow (Q \vee P)$ | add sentence lem1 |
| 6 | $(P \rightarrow (Q \vee P)) \rightarrow ((A \vee P) \rightarrow (A \vee (Q \vee P)))$ | replace $Q$ by $Q \vee P$ in 1 |
| 7 | $(A \vee P) \rightarrow (A \vee (Q \vee P))$ | MP with 5, 6 |
| 8 | $((A \vee P) \rightarrow P_8) \rightarrow ((P_9 \vee (A \vee P)) \rightarrow (P_9 \vee P_8))$ | replace $P_7$ by $A \vee P$ in 4 |
| 9 | $((A \vee P) \rightarrow (A \vee (Q \vee P))) \rightarrow ((P_9 \vee (A \vee P)) \rightarrow (P_9 \vee (A \vee (Q \vee P))))$ | replace $P_8$ by $A \vee (Q \vee P)$ in 8 |
| 10 | $(P_9 \vee (A \vee P)) \rightarrow (P_9 \vee (A \vee (Q \vee P)))$ | MP with 7, 9 |
| 11 | $(Q \vee (A \vee P)) \rightarrow (Q \vee (A \vee (Q \vee P)))$ | replace $P_9$ by $Q$ in 10 |
| 12 | $(P \vee Q) \rightarrow (Q \vee P)$ | add axiom axiom3 |
| 13 | $(D \vee Q) \rightarrow (Q \vee D)$ | replace $P$ by $D$ in 12 |
| 14 | $(D \vee (A \vee (Q \vee P))) \rightarrow ((A \vee (Q \vee P)) \vee D)$ | replace $Q$ by $A \vee (Q \vee P)$ in 13 |
| 15 | $(Q \vee (A \vee (Q \vee P))) \rightarrow ((A \vee (Q \vee P)) \vee Q)$ | replace $D$ by $Q$ in 14 |
| 16 | $((Q \vee (A \vee (Q \vee P))) \rightarrow P_8) \rightarrow ((P_9 \vee (Q \vee (A \vee (Q \vee P)))) \rightarrow (P_9 \vee P_8))$ | replace $P_7$ by $Q \vee (A \vee (Q \vee P))$ in 4 |
| 17 | $((Q \vee (A \vee (Q \vee P))) \rightarrow ((A \vee (Q \vee P)) \vee Q)) \rightarrow ((P_9 \vee (Q \vee (A \vee (Q \vee P)))) \rightarrow (P_9 \vee ((A \vee (Q \vee P)) \vee Q)))$ | replace $P_8$ by $(A \vee (Q \vee P)) \vee Q$ in 16 |
| 18 | $(P_9 \vee (Q \vee (A \vee (Q \vee P)))) \rightarrow (P_9 \vee ((A \vee (Q \vee P)) \vee Q))$ | MP with 15, 17 |
| 19 | $(\neg(Q \vee (A \vee P)) \vee (Q \vee (A \vee (Q \vee P)))) \rightarrow (\neg(Q \vee (A \vee P)) \vee ((A \vee (Q \vee P)) \vee Q))$ | replace $P_9$ by $\neg(Q \vee (A \vee P))$ in 18 |
| 20 | $((Q \vee (A \vee P)) \rightarrow (Q \vee (A \vee (Q \vee P)))) \rightarrow (\neg(Q \vee (A \vee P)) \vee ((A \vee (Q \vee P)) \vee Q))$ | reverse abbreviation impl in 19 at occurence 1 |
| 21 | $((Q \vee (A \vee P)) \rightarrow (Q \vee (A \vee (Q \vee P)))) \rightarrow ((Q \vee (A \vee P)) \rightarrow ((A \vee (Q \vee P)) \vee Q))$ | reverse abbreviation impl in 20 at occurence 1 |
| 22 | $(Q \vee (A \vee P)) \rightarrow ((A \vee (Q \vee P)) \vee Q)$ | MP with 11, 21 |
| 23 | $P \rightarrow (P \vee Q)$ | add axiom axiom2 |
| 24 | $A \rightarrow (A \vee Q)$ | replace $P$ by $A$ in 23 |
| 25 | $A \rightarrow (A \vee P)$ | replace $Q$ by $P$ in 24 |
| 26 | $Q \rightarrow (Q \vee P)$ | replace $A$ by $Q$ in 25 |
| 27 | $P \rightarrow (A \vee P)$ | replace $Q$ by $A$ in 5 |
| 28 | $(Q \vee P) \rightarrow (A \vee (Q \vee P))$ | replace $P$ by $Q \vee P$ in 27 |
| 29 | $((Q \vee P) \rightarrow P_8) \rightarrow ((P_9 \vee (Q \vee P)) \rightarrow (P_9 \vee P_8))$ | replace $P_7$ by $Q \vee P$ in 4 |
| 30 | $((Q \vee P) \rightarrow (A \vee (Q \vee P))) \rightarrow ((P_9 \vee (Q \vee P)) \rightarrow (P_9 \vee (A \vee (Q \vee P))))$ | replace $P_8$ by $A \vee (Q \vee P)$ in 29 |
| 31 | $(P_9 \vee (Q \vee P)) \rightarrow (P_9 \vee (A \vee (Q \vee P)))$ | MP with 28, 30 |

| | | |
|---|---|---|
| 32 | $(\neg Q \vee (Q \vee P)) \to (\neg Q \vee (A \vee (Q \vee P)))$ | replace $P_9$ by $\neg Q$ in 31 |
| 33 | $(Q \to (Q \vee P)) \to (\neg Q \vee (A \vee (Q \vee P)))$ | reverse abbreviation impl in 32 at occurence 1 |
| 34 | $(Q \to (Q \vee P)) \to (Q \to (A \vee (Q \vee P)))$ | reverse abbreviation impl in 33 at occurence 1 |
| 35 | $Q \to (A \vee (Q \vee P))$ | MP with 26, 34 |
| 36 | $(Q \to P_8) \to ((P_9 \vee Q) \to (P_9 \vee P_8))$ | replace $P_7$ by $Q$ in 4 |
| 37 | $(Q \to (A \vee (Q \vee P))) \to ((P_9 \vee Q) \to (P_9 \vee (A \vee (Q \vee P))))$ | replace $P_8$ by $A \vee (Q \vee P)$ in 36 |
| 38 | $(P_9 \vee Q) \to (P_9 \vee (A \vee (Q \vee P)))$ | MP with 35, 37 |
| 39 | $((A \vee (Q \vee P)) \vee Q) \to ((A \vee (Q \vee P)) \vee (A \vee (Q \vee P)))$ | replace $P_9$ by $A \vee (Q \vee P)$ in 38 |
| 40 | $(P \vee P) \to P$ | add axiom axiom1 |
| 41 | $((A \vee (Q \vee P)) \vee (A \vee (Q \vee P))) \to (A \vee (Q \vee P))$ | replace $P$ by $A \vee (Q \vee P)$ in 40 |
| 42 | $(((A \vee (Q \vee P)) \vee (A \vee (Q \vee P))) \to P_8) \to ((P_9 \vee ((A \vee (Q \vee P)) \vee (A \vee (Q \vee P)))) \to (P_9 \vee P_8))$ | replace $P_7$ by $(A \vee (Q \vee P)) \vee (A \vee (Q \vee P))$ in 4 |
| 43 | $(((A \vee (Q \vee P)) \vee (A \vee (Q \vee P))) \to (A \vee (Q \vee P))) \to ((P_9 \vee ((A \vee (Q \vee P)) \vee (A \vee (Q \vee P)))) \to (P_9 \vee (A \vee (Q \vee P))))$ | replace $P_8$ by $A \vee (Q \vee P)$ in 42 |
| 44 | $(P_9 \vee ((A \vee (Q \vee P)) \vee (A \vee (Q \vee P)))) \to (P_9 \vee (A \vee (Q \vee P)))$ | MP with 41, 43 |
| 45 | $(\neg((A \vee (Q \vee P)) \vee Q) \vee ((A \vee (Q \vee P)) \vee (A \vee (Q \vee P)))) \to (\neg((A \vee (Q \vee P)) \vee Q) \vee (A \vee (Q \vee P)))$ | replace $P_9$ by $\neg((A \vee (Q \vee P)) \vee Q)$ in 44 |
| 46 | $(((A \vee (Q \vee P)) \vee Q) \to ((A \vee (Q \vee P)) \vee (A \vee (Q \vee P)))) \to (\neg((A \vee (Q \vee P)) \vee Q) \vee (A \vee (Q \vee P)))$ | reverse abbreviation impl in 45 at occurence 1 |
| 47 | $(((A \vee (Q \vee P)) \vee Q) \to ((A \vee (Q \vee P)) \vee (A \vee (Q \vee P)))) \to (((A \vee (Q \vee P)) \vee Q) \to (A \vee (Q \vee P)))$ | reverse abbreviation impl in 46 at occurence 1 |
| 48 | $((A \vee (Q \vee P)) \vee Q) \to (A \vee (Q \vee P))$ | MP with 39, 47 |
| 49 | $(((A \vee (Q \vee P)) \vee Q) \to P_8) \to ((P_9 \vee ((A \vee (Q \vee P)) \vee Q)) \to (P_9 \vee P_8))$ | replace $P_7$ by $(A \vee (Q \vee P)) \vee Q$ in 4 |
| 50 | $(((A \vee (Q \vee P)) \vee Q) \to (A \vee (Q \vee P))) \to ((P_9 \vee ((A \vee (Q \vee P)) \vee Q)) \to (P_9 \vee (A \vee (Q \vee P))))$ | replace $P_8$ by $A \vee (Q \vee P)$ in 49 |
| 51 | $(P_9 \vee ((A \vee (Q \vee P)) \vee Q)) \to (P_9 \vee (A \vee (Q \vee P)))$ | MP with 48, 50 |
| 52 | $(\neg(Q \vee (A \vee P)) \vee ((A \vee (Q \vee P)) \vee Q)) \to (\neg(Q \vee (A \vee P)) \vee (A \vee (Q \vee P)))$ | replace $P_9$ by $\neg(Q \vee (A \vee P))$ in 51 |
| 53 | $((Q \vee (A \vee P)) \to ((A \vee (Q \vee P)) \vee Q)) \to (\neg(Q \vee (A \vee P)) \vee (A \vee (Q \vee P)))$ | reverse abbreviation impl in 52 at occurence 1 |
| 54 | $((Q \vee (A \vee P)) \to ((A \vee (Q \vee P)) \vee Q)) \to ((Q \vee (A \vee P)) \to (A \vee (Q \vee P)))$ | reverse abbreviation impl in 53 at occurence 1 |
| 55 | $(Q \vee (A \vee P)) \to (A \vee (Q \vee P))$ | MP with 22, 54 |
| 56 | $(P_7 \vee (A \vee P)) \to (A \vee (P_7 \vee P))$ | replace $Q$ by $P_7$ in 55 |
| 57 | $(P_7 \vee (Q \vee P)) \to (Q \vee (P_7 \vee P))$ | replace $A$ by $Q$ in 56 |
| 58 | $(P_7 \vee (Q \vee A)) \to (Q \vee (P_7 \vee A))$ | replace $P$ by $A$ in 57 |
| 59 | $(P \vee (Q \vee A)) \to (Q \vee (P \vee A))$ | replace $P_7$ by $P$ in 58 |

□

The fifth primitive proposition is our fourth axiom:

**Theorem 0.6 (prin5).**

$$(P \to Q) \to ((A \vee P) \to (A \vee Q))$$

*Proof.*

1 $(P \rightarrow Q) \rightarrow ((A \vee P) \rightarrow (A \vee Q))$ add axiom axiom4

□