First theorems of Propositional Calculus

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Abstract

This module includes first proofs of propositional calculus theorems. The following theorems and proofs are adapted from D. Hilbert and W. Ackermann's 'Grundzuege der theoretischen Logik' (Berlin 1928, Springer)

Specification

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Content

First we prove a simple implication, that follows directly from the fourth axiom:

Theorem 0.1 (hilb1).

$$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$$

Proof.

1	$(P \rightarrow Q) \rightarrow ((A \vee P) \rightarrow (A \vee Q))$	add axiom axiom4
2	$(P \rightarrow Q) \rightarrow ((\neg A \vee P) \rightarrow (\neg A \vee Q))$	replace A by $\neg A$ in 1
3	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (\neg A \vee Q))$	reverse abbreviation impl in 2 at occurence 1
4	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$	reverse abbreviation impl in 3 at occurence 1

□

This proposition is the form for the Hypothetical Syllogism.

The self implication could be derived:

Theorem 0.2 (hilb2).

$$P \rightarrow P$$

Proof.

1	$P \rightarrow (P \vee Q)$	add axiom axiom2
2	$P \rightarrow (P \vee P)$	replace Q by P in 1
3	$(P \vee P) \rightarrow P$	add axiom axiom1
4	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$	add sentence hilb1
5	$(P \rightarrow Q) \rightarrow ((B \rightarrow P) \rightarrow (B \rightarrow Q))$	replace A by B in 4
6	$(P \rightarrow C) \rightarrow ((B \rightarrow P) \rightarrow (B \rightarrow C))$	replace Q by C in 5
7	$(D \rightarrow C) \rightarrow ((B \rightarrow D) \rightarrow (B \rightarrow C))$	replace P by D in 6
8	$(D \rightarrow C) \rightarrow ((P \rightarrow D) \rightarrow (P \rightarrow C))$	replace B by P in 7
9	$(D \rightarrow P) \rightarrow ((P \rightarrow D) \rightarrow (P \rightarrow P))$	replace C by P in 8
10	$((P \vee P) \rightarrow P) \rightarrow ((P \rightarrow (P \vee P)) \rightarrow (P \rightarrow P))$	replace D by $P \vee P$ in 9
11	$(P \rightarrow (P \vee P)) \rightarrow (P \rightarrow P)$	MP with 3, 10
12	$P \rightarrow P$	MP with 2, 11

□

One form of the classical 'tertium non datur'

Theorem 0.3 (hilb3).

$$\neg P \vee P$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$\neg P \vee P$	use abbreviation impl in 1 at occurence 1

□

The standard form of the excluded middle:

Theorem 0.4 (hilb4).

$$P \lor \neg P$$

Proof.

1	$\neg P \lor P$	add sentence hilb3
2	$(P \lor Q) \to (Q \lor P)$	add axiom axiom3
3	$(P \lor A) \to (A \lor P)$	replace Q by A in 2
4	$(B \lor A) \to (A \lor B)$	replace P by B in 3
5	$(B \lor P) \to (P \lor B)$	replace A by P in 4
6	$(\neg P \lor P) \to (P \lor \neg P)$	replace B by $\neg P$ in 5
7	$P \lor \neg P$	MP with 1, 6

□

Double negation is implicated:

Theorem 0.5 (hilb5).

$$P \to \neg\neg P$$

Proof.

1	$P \lor \neg P$	add sentence hilb4
2	$\neg P \lor \neg\neg P$	replace P by $\neg P$ in 1
3	$P \to \neg\neg P$	reverse abbreviation impl in 2 at occurence 1

□

The reverse is also true:

Theorem 0.6 (hilb6).

$$\neg\neg P \to P$$

Proof.

1	$P \to \neg\neg P$	add sentence hilb5
2	$\neg P \to \neg\neg\neg P$	replace P by $\neg P$ in 1
3	$(P \to Q) \to ((A \lor P) \to (A \lor Q))$	add axiom axiom4
4	$(P \to Q) \to ((B \lor P) \to (B \lor Q))$	replace A by B in 3
5	$(P \to C) \to ((B \lor P) \to (B \lor C))$	replace Q by C in 4
6	$(D \to C) \to ((B \lor D) \to (B \lor C))$	replace P by D in 5
7	$(D \to C) \to ((P \lor D) \to (P \lor C))$	replace B by P in 6
8	$(D \to \neg\neg\neg P) \to ((P \lor D) \to (P \lor \neg\neg\neg P))$	replace C by $\neg\neg\neg P$ in 7
9	$(\neg P \to \neg\neg\neg P) \to ((P \lor \neg P) \to (P \lor \neg\neg\neg P))$	replace D by $\neg P$ in 8
10	$(P \lor \neg P) \to (P \lor \neg\neg\neg P)$	MP with 2, 9
11	$P \lor \neg P$	add sentence hilb4
12	$P \lor \neg\neg\neg P$	MP with 11, 10
13	$(P \lor Q) \to (Q \lor P)$	add axiom axiom3
14	$(P \lor A) \to (A \lor P)$	replace Q by A in 13
15	$(B \lor A) \to (A \lor B)$	replace P by B in 14
16	$(B \lor \neg\neg\neg P) \to (\neg\neg\neg P \lor B)$	replace A by $\neg\neg\neg P$ in 15
17	$(P \lor \neg\neg\neg P) \to (\neg\neg\neg P \lor P)$	replace B by P in 16

18	$\neg\neg\neg P \lor P$	MP with 12, 17
19	$\neg\neg P \to P$	reverse abbreviation impl in 18 at occurence 1

□

The correct reverse of an implication:

Theorem 0.7 (hilb7).

$$(P \to Q) \to (\neg Q \to \neg P)$$

Proof.

1	$P \to \neg\neg P$	add sentence hilb5
2	$Q \to \neg\neg Q$	replace P by Q in 1
3	$(P \to Q) \to ((A \lor P) \to (A \lor Q))$	add axiom axiom4
4	$(P \to Q) \to ((B \lor P) \to (B \lor Q))$	replace A by B in 3
5	$(P \to C) \to ((B \lor P) \to (B \lor C))$	replace Q by C in 4
6	$(D \to C) \to ((B \lor D) \to (B \lor C))$	replace P by D in 5
7	$(D \to C) \to ((\neg P \lor D) \to (\neg P \lor C))$	replace B by $\neg P$ in 6
8	$(D \to \neg\neg Q) \to ((\neg P \lor D) \to (\neg P \lor \neg\neg Q))$	replace C by $\neg\neg Q$ in 7
9	$(Q \to \neg\neg Q) \to ((\neg P \lor Q) \to (\neg P \lor \neg\neg Q))$	replace D by Q in 8
10	$(\neg P \lor Q) \to (\neg P \lor \neg\neg Q)$	MP with 2, 9
11	$(P \lor Q) \to (Q \lor P)$	add axiom axiom3
12	$(P \lor A) \to (A \lor P)$	replace Q by A in 11
13	$(B \lor A) \to (A \lor B)$	replace P by B in 12
14	$(B \lor \neg\neg Q) \to (\neg\neg Q \lor B)$	replace A by $\neg\neg Q$ in 13
15	$(\neg P \lor \neg\neg Q) \to (\neg\neg Q \lor \neg P)$	replace B by $\neg P$ in 14
16	$(P \to Q) \to ((A \to P) \to (A \to Q))$	add sentence hilb1
17	$(P \to Q) \to ((B \to P) \to (B \to Q))$	replace A by B in 16
18	$(P \to C) \to ((B \to P) \to (B \to C))$	replace Q by C in 17
19	$(D \to C) \to ((B \to D) \to (B \to C))$	replace P by D in 18
20	$(D \to C) \to (((\neg P \lor Q) \to D) \to ((\neg P \lor Q) \to C))$	replace B by $\neg P \lor Q$ in 19
21	$(D \to (\neg\neg Q \lor \neg P)) \to (((\neg P \lor Q) \to D) \to ((\neg P \lor Q) \to (\neg\neg Q \lor \neg P)))$	replace C by $\neg\neg Q \lor \neg P$ in 20
22	$((\neg P \lor \neg\neg Q) \to (\neg\neg Q \lor \neg P)) \to (((\neg P \lor Q) \to (\neg P \lor \neg\neg Q)) \to ((\neg P \lor Q) \to (\neg\neg Q \lor \neg P)))$	replace D by $\neg P \lor \neg\neg Q$ in 21
23	$((\neg P \lor Q) \to (\neg P \lor \neg\neg Q)) \to ((\neg P \lor Q) \to (\neg\neg Q \lor \neg P))$	MP with 15, 22
24	$(\neg P \lor Q) \to (\neg\neg Q \lor \neg P)$	MP with 10, 23
25	$(P \to Q) \to (\neg\neg Q \lor \neg P)$	reverse abbreviation impl in 24 at occurence 1
26	$(P \to Q) \to (\neg Q \to \neg P)$	reverse abbreviation impl in 25 at occurence 1

□

Definition of an Implication 1. part:

Theorem 0.8 (defimpl1).

$$(P \to Q) \to (\neg P \lor Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$A \rightarrow A$	replace P by A in 1
3	$(P \rightarrow Q) \rightarrow (P \rightarrow Q)$	replace A by $P \rightarrow Q$ in 2
4	$(P \rightarrow Q) \rightarrow (\neg P \vee Q)$	use abbreviation impl in 3 at occurence 3

□

Definition of an Implication 2. part:

Theorem 0.9 (defimpl2).

$$(\neg P \vee Q) \rightarrow (P \rightarrow Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$A \rightarrow A$	replace P by A in 1
3	$(P \rightarrow Q) \rightarrow (P \rightarrow Q)$	replace A by $P \rightarrow Q$ in 2
4	$(\neg P \vee Q) \rightarrow (P \rightarrow Q)$	use abbreviation impl in 3 at occurence 2

□

Definition of a Conjunction 1. part:

Theorem 0.10 (defand1).

$$(P \wedge Q) \rightarrow \neg(\neg P \vee \neg Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$A \rightarrow A$	replace P by A in 1
3	$(P \wedge Q) \rightarrow (P \wedge Q)$	replace A by $P \wedge Q$ in 2
4	$(P \wedge Q) \rightarrow \neg(\neg P \vee \neg Q)$	use abbreviation and in 3 at occurence 2

□

Definition of a Conjunction 2. part:

Theorem 0.11 (defand2).

$$\neg(\neg P \vee \neg Q) \rightarrow (P \wedge Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$A \rightarrow A$	replace P by A in 1
3	$(P \wedge Q) \rightarrow (P \wedge Q)$	replace A by $P \wedge Q$ in 2
4	$\neg(\neg P \vee \neg Q) \rightarrow (P \wedge Q)$	use abbreviation and in 3 at occurence 1

□

Definition of an Equivalence 1. part:

Theorem 0.12 (defequi1).

$$(P \leftrightarrow Q) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P))$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$A \rightarrow A$	replace P by A in 1
3	$(P \leftrightarrow Q) \rightarrow (P \leftrightarrow Q)$	replace A by $P \leftrightarrow Q$ in 2
4	$(P \leftrightarrow Q) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P))$	use abbreviation equi in 3 at occurence 2

□

Definition of an Equivalence 2. part:

Theorem 0.13 (defequi2).

$$((P \rightarrow Q) \wedge (Q \rightarrow P)) \rightarrow (P \leftrightarrow Q)$$

Proof.

1	$P \rightarrow P$	add sentence hilb2
2	$A \rightarrow A$	replace P by A in 1
3	$(P \leftrightarrow Q) \rightarrow (P \leftrightarrow Q)$	replace A by $P \leftrightarrow Q$ in 2
4	$((P \rightarrow Q) \wedge (Q \rightarrow P)) \rightarrow (P \leftrightarrow Q)$	use abbreviation equi in 3 at occurence 1

□

A simular formulation for the second axiom:

Theorem 0.14 (hilb8).

$$P \rightarrow (Q \vee P)$$

Proof.

1	$P \rightarrow (P \vee Q)$	add axiom axiom2
2	$(P \vee Q) \rightarrow (Q \vee P)$	add axiom axiom3
3	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$	add sentence hilb1
4	$(P \rightarrow Q) \rightarrow ((B \rightarrow P) \rightarrow (B \rightarrow Q))$	replace A by B in 3
5	$(P \rightarrow C) \rightarrow ((B \rightarrow P) \rightarrow (B \rightarrow C))$	replace Q by C in 4
6	$(D \rightarrow C) \rightarrow ((B \rightarrow D) \rightarrow (B \rightarrow C))$	replace P by D in 5
7	$(D \rightarrow C) \rightarrow ((P \rightarrow D) \rightarrow (P \rightarrow C))$	replace B by P in 6
8	$(D \rightarrow (Q \vee P)) \rightarrow ((P \rightarrow D) \rightarrow (P \rightarrow (Q \vee P)))$	replace C by $Q \vee P$ in 7
9	$((P \vee Q) \rightarrow (Q \vee P)) \rightarrow ((P \rightarrow (P \vee Q)) \rightarrow (P \rightarrow (Q \vee P)))$	replace D by $P \vee Q$ in 8
10	$(P \rightarrow (P \vee Q)) \rightarrow (P \rightarrow (Q \vee P))$	MP with 2, 9
11	$P \rightarrow (Q \vee P)$	MP with 1, 10

□

A technical lemma (equal to the third axiom):

Theorem 0.15 (hilb9).

$$(P \vee Q) \rightarrow (Q \vee P)$$

Proof.

1	$(P \vee Q) \rightarrow (Q \vee P)$	add axiom axiom3

□

And another technical lemma (simular to the third axiom):

Theorem 0.16 (hilb10).

$$(Q \vee P) \rightarrow (P \vee Q)$$

Proof.

1	$(P \vee Q) \rightarrow (Q \vee P)$	add axiom axiom3
2	$(P \vee A) \rightarrow (A \vee P)$	replace Q by A in 1
3	$(B \vee A) \rightarrow (A \vee B)$	replace P by B in 2
4	$(B \vee P) \rightarrow (P \vee B)$	replace A by P in 3
5	$(Q \vee P) \rightarrow (P \vee Q)$	replace B by Q in 4

□

A technical lemma (equal to the first axiom):

Theorem 0.17 (hilb11).

$$(P \vee P) \rightarrow P$$

Proof.

1	$(P \vee P) \rightarrow P$	add axiom axiom1

□

A simple propositon that follows directly from the second axiom:

Theorem 0.18 (hilb12).

$$P \rightarrow (P \vee P)$$

Proof.

1	$P \rightarrow (P \vee Q)$	add axiom axiom2
2	$P \rightarrow (P \vee P)$	replace Q by P in 1

□

This is a pre form for the associative law:

Theorem 0.19 (hilb13).

$$(P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A))$$

Proof.

1	$P \rightarrow (Q \vee P)$	add sentence hilb8

2	$P \to (B \vee P)$	replace Q by B in 1
3	$C \to (B \vee C)$	replace P by C in 2
4	$C \to (P \vee C)$	replace B by P in 3
5	$A \to (P \vee A)$	replace C by A in 4
6	$(P \to Q) \to ((A \vee P) \to (A \vee Q))$	add axiom axiom4
7	$(P \to Q) \to ((B \vee P) \to (B \vee Q))$	replace A by B in 6
8	$(P \to C) \to ((B \vee P) \to (B \vee C))$	replace Q by C in 7
9	$(D \to C) \to ((B \vee D) \to (B \vee C))$	replace P by D in 8
10	$(D \to C) \to ((Q \vee D) \to (Q \vee C))$	replace B by Q in 9
11	$(D \to (P \vee A)) \to ((Q \vee D) \to (Q \vee (P \vee A)))$	replace C by $P \vee A$ in 10
12	$(A \to (P \vee A)) \to ((Q \vee A) \to (Q \vee (P \vee A)))$	replace D by A in 11
13	$(Q \vee A) \to (Q \vee (P \vee A))$	MP with 5, 12
14	$(D \to C) \to ((P \vee D) \to (P \vee C))$	replace B by P in 9
15	$(D \to (Q \vee (P \vee A))) \to ((P \vee D) \to (P \vee (Q \vee (P \vee A))))$	replace C by $Q \vee (P \vee A)$ in 14
16	$((Q \vee A) \to (Q \vee (P \vee A))) \to ((P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A))))$	replace D by $Q \vee A$ in 15
17	$(P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A)))$	MP with 13, 16
18	$(P \vee Q) \to (Q \vee P)$	add sentence hilb9
19	$(P \vee B) \to (B \vee P)$	replace Q by B in 18
20	$(C \vee B) \to (B \vee C)$	replace P by C in 19
21	$(C \vee (Q \vee (P \vee A))) \to ((Q \vee (P \vee A)) \vee C)$	replace B by $Q \vee (P \vee A)$ in 20
22	$(P \vee (Q \vee (P \vee A))) \to ((Q \vee (P \vee A)) \vee P)$	replace C by P in 21
23	$(P \to Q) \to (\neg P \vee Q)$	add sentence defimpl1
24	$(\neg P \vee Q) \to (P \to Q)$	add sentence defimpl2
25	$(D \to C) \to ((\neg(P \vee (Q \vee A)) \vee D) \to (\neg(P \vee (Q \vee A)) \vee C))$	replace B by $\neg(P \vee (Q \vee A))$ in 9
26	$(D \to ((Q \vee (P \vee A)) \vee P)) \to ((\neg(P \vee (Q \vee A)) \vee D) \to (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)))$	replace C by $(Q \vee (P \vee A)) \vee P$ in 25
27	$((P \vee (Q \vee (P \vee A))) \to ((Q \vee (P \vee A)) \vee P)) \to ((\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \to (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)))$	replace D by $P \vee (Q \vee (P \vee A))$ in 26
28	$(\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \to (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))$	MP with 22, 27
29	$(P \to B) \to (\neg P \vee B)$	replace Q by B in 23
30	$(C \to B) \to (\neg C \vee B)$	replace P by C in 29
31	$(C \to (P \vee (Q \vee (P \vee A)))) \to (\neg C \vee (P \vee (Q \vee (P \vee A))))$	replace B by $P \vee (Q \vee (P \vee A))$ in 30
32	$((P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A)))) \to (\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A))))$	replace C by $P \vee (Q \vee A)$ in 31
33	$(P \to Q) \to ((A \to P) \to (A \to Q))$	add sentence hilb1
34	$(P \to Q) \to ((B \to P) \to (B \to Q))$	replace A by B in 33
35	$(P \to C) \to ((B \to P) \to (B \to C))$	replace Q by C in 34
36	$(D \to C) \to ((B \to D) \to (B \to C))$	replace P by D in 35
37	$(D \to C) \to ((((P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A)))) \to D) \to (((P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A)))) \to C))$	replace B by $(P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A)))$ in 36
38	$(D \to (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))) \to ((((P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A)))) \to D) \to (((P \vee (Q \vee A)) \to (P \vee (Q \vee (P \vee A)))) \to (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))))$	replace C by $\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)$ in 37

39	$((\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))) \rightarrow ((((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A))))) \rightarrow (((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))))$	replace D by $\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A)))$ in 38
40	$(((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee (P \vee (Q \vee (P \vee A))))) \rightarrow (((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)))$	MP with 28, 39
41	$((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))$	MP with 32, 40
42	$(\neg P \vee B) \rightarrow (P \rightarrow B)$	replace Q by B in 24
43	$(\neg C \vee B) \rightarrow (C \rightarrow B)$	replace P by C in 42
44	$(\neg C \vee ((Q \vee (P \vee A)) \vee P)) \rightarrow (C \rightarrow ((Q \vee (P \vee A)) \vee P))$	replace B by $(Q \vee (P \vee A)) \vee P$ in 43
45	$(\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \rightarrow ((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P))$	replace C by $P \vee (Q \vee A)$ in 44
46	$(D \rightarrow ((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P))) \rightarrow ((((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow D) \rightarrow (((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow ((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P))))$	replace C by $(P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P)$ in 37
47	$((\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)) \rightarrow ((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P))) \rightarrow ((((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))) \rightarrow (((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow ((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P))))$	replace D by $\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P)$ in 46
48	$(((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow (\neg(P \vee (Q \vee A)) \vee ((Q \vee (P \vee A)) \vee P))) \rightarrow (((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow ((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P)))$	MP with 45, 47
49	$((P \vee (Q \vee A)) \rightarrow (P \vee (Q \vee (P \vee A)))) \rightarrow ((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P))$	MP with 41, 48
50	$(P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P)$	MP with 17, 49
51	$(P \vee A) \rightarrow (Q \vee (P \vee A))$	replace P by $P \vee A$ in 1
52	$P \rightarrow (P \vee Q)$	add axiom axiom2
53	$P \rightarrow (P \vee A)$	replace Q by A in 52
54	$(D \rightarrow C) \rightarrow ((P \rightarrow D) \rightarrow (P \rightarrow C))$	replace B by P in 36
55	$(D \rightarrow (Q \vee (P \vee A))) \rightarrow ((P \rightarrow D) \rightarrow (P \rightarrow (Q \vee (P \vee A))))$	replace C by $Q \vee (P \vee A)$ in 54
56	$((P \vee A) \rightarrow (Q \vee (P \vee A))) \rightarrow ((P \rightarrow (P \vee A)) \rightarrow (P \rightarrow (Q \vee (P \vee A))))$	replace D by $P \vee A$ in 55
57	$(P \rightarrow (P \vee A)) \rightarrow (P \rightarrow (Q \vee (P \vee A)))$	MP with 51, 56
58	$P \rightarrow (Q \vee (P \vee A))$	MP with 53, 57
59	$(D \rightarrow C) \rightarrow (((Q \vee (P \vee A)) \vee D) \rightarrow ((Q \vee (P \vee A)) \vee C))$	replace B by $Q \vee (P \vee A)$ in 9
60	$(D \rightarrow (Q \vee (P \vee A))) \rightarrow (((Q \vee (P \vee A)) \vee D) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))$	replace C by $Q \vee (P \vee A)$ in 59
61	$(P \rightarrow (Q \vee (P \vee A))) \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))$	replace D by P in 60
62	$((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$	MP with 58, 61

63	$(P \vee P) \rightarrow P$	add sentence hilb11
64	$(B \vee B) \rightarrow B$	replace P by B in 63
65	$((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))) \rightarrow (Q \vee (P \vee A))$	replace B by $Q \vee (P \vee A)$ in 64
66	$(D \rightarrow C) \rightarrow ((\neg((Q \vee (P \vee A)) \vee P) \vee D) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee C))$	replace B by $\neg((Q \vee (P \vee A)) \vee P)$ in 9
67	$(D \rightarrow (Q \vee (P \vee A))) \rightarrow ((\neg((Q \vee (P \vee A)) \vee P) \vee D) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))))$	replace C by $Q \vee (P \vee A)$ in 66
68	$(((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))) \rightarrow (Q \vee (P \vee A))) \rightarrow ((\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))))$	replace D by $(Q \vee (P \vee A)) \vee (Q \vee (P \vee A))$ in 67
69	$(\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))$	MP with 65, 68
70	$(C \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg C \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))$	replace B by $(Q \vee (P \vee A)) \vee (Q \vee (P \vee A))$ in 30
71	$(((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))$	replace C by $(Q \vee (P \vee A)) \vee P$ in 70
72	$(D \rightarrow C) \rightarrow (((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow D) \rightarrow ((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow C))$	replace B by $((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$ in 36
73	$(D \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \rightarrow (((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow D) \rightarrow ((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))))$	replace C by $\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))$ in 72
74	$((\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \rightarrow (((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))) \rightarrow ((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))))$	replace D by $\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))$ in 73
75	$((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A))))) \rightarrow ((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))))$	MP with 69, 74
76	$(((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))$	MP with 71, 75
77	$(\neg C \vee (Q \vee (P \vee A))) \rightarrow (C \rightarrow (Q \vee (P \vee A)))$	replace B by $Q \vee (P \vee A)$ in 43
78	$(\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))) \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A)))$	replace C by $(Q \vee (P \vee A)) \vee P$ in 77
79	$(D \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A)))) \rightarrow (((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow D) \rightarrow ((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A)))))$	replace C by $((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A))$ in 72

80	$((\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))) \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A)))) \rightarrow ((((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \rightarrow ((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A)))))$	replace D by $\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A))$ in 79
81	$((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (\neg((Q \vee (P \vee A)) \vee P) \vee (Q \vee (P \vee A)))) \rightarrow ((((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A))))$	MP with 78, 80
82	$(((Q \vee (P \vee A)) \vee P) \rightarrow ((Q \vee (P \vee A)) \vee (Q \vee (P \vee A)))) \rightarrow (((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A)))$	MP with 76, 81
83	$((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A))$	MP with 62, 82
84	$(D \rightarrow C) \rightarrow (((P \vee (Q \vee A)) \rightarrow D) \rightarrow ((P \vee (Q \vee A)) \rightarrow C))$	replace B by $P \vee (Q \vee A)$ in 36
85	$(D \rightarrow (Q \vee (P \vee A))) \rightarrow (((P \vee (Q \vee A)) \rightarrow D) \rightarrow ((P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A))))$	replace C by $Q \vee (P \vee A)$ in 84
86	$(((Q \vee (P \vee A)) \vee P) \rightarrow (Q \vee (P \vee A))) \rightarrow (((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P)) \rightarrow ((P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A))))$	replace D by $(Q \vee (P \vee A)) \vee P$ in 85
87	$((P \vee (Q \vee A)) \rightarrow ((Q \vee (P \vee A)) \vee P)) \rightarrow ((P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A)))$	MP with 83, 86
88	$(P \vee (Q \vee A)) \rightarrow (Q \vee (P \vee A))$	MP with 50, 87

□

The associative law for the disjunction (first direction):

Theorem 0.20 (hilb14).

$$(P \vee (Q \vee A)) \rightarrow ((P \vee Q) \vee A)$$

Proof.

1	$(P \vee Q) \rightarrow (Q \vee P)$	add sentence hilb9
2	$(P \vee B) \rightarrow (B \vee P)$	replace Q by B in 1
3	$(C \vee B) \rightarrow (B \vee C)$	replace P by C in 2
4	$(C \vee A) \rightarrow (A \vee C)$	replace B by A in 3
5	$(Q \vee A) \rightarrow (A \vee Q)$	replace C by Q in 4
6	$(P \rightarrow Q) \rightarrow ((A \vee P) \rightarrow (A \vee Q))$	add axiom axiom4
7	$(P \rightarrow Q) \rightarrow ((B \vee P) \rightarrow (B \vee Q))$	replace A by B in 6
8	$(P \rightarrow C) \rightarrow ((B \vee P) \rightarrow (B \vee C))$	replace Q by C in 7
9	$(D \rightarrow C) \rightarrow ((B \vee D) \rightarrow (B \vee C))$	replace P by D in 8
10	$(D \rightarrow C) \rightarrow ((P \vee D) \rightarrow (P \vee C))$	replace B by P in 9
11	$(D \rightarrow (A \vee Q)) \rightarrow ((P \vee D) \rightarrow (P \vee (A \vee Q)))$	replace C by $A \vee Q$ in 10
12	$((Q \vee A) \rightarrow (A \vee Q)) \rightarrow ((P \vee (Q \vee A)) \rightarrow (P \vee (A \vee Q)))$	replace D by $Q \vee A$ in 11
13	$(P \vee (Q \vee A)) \rightarrow (P \vee (A \vee Q))$	MP with 5, 12
14	$(P \rightarrow Q) \rightarrow (\neg P \vee Q)$	add sentence defimpl1
15	$(\neg P \vee Q) \rightarrow (P \rightarrow Q)$	add sentence defimpl2
16	$(P \rightarrow B) \rightarrow (\neg P \vee B)$	replace Q by B in 14
17	$(C \rightarrow B) \rightarrow (\neg C \vee B)$	replace P by C in 16
18	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$	add sentence hilb1
19	$(P \rightarrow Q) \rightarrow ((B \rightarrow P) \rightarrow (B \rightarrow Q))$	replace A by B in 18

20	$(P \to C) \to ((B \to P) \to (B \to C))$	replace Q by C in 19
21	$(D \to C) \to ((B \to D) \to (B \to C))$	replace P by D in 20
22	$(\neg P \vee B) \to (P \to B)$	replace Q by B in 15
23	$(\neg C \vee B) \to (C \to B)$	replace P by C in 22
24	$(P \vee (Q \vee A)) \to (Q \vee (P \vee A))$	add sentence hilb13
25	$(P \vee (Q \vee B)) \to (Q \vee (P \vee B))$	replace A by B in 24
26	$(P \vee (C \vee B)) \to (C \vee (P \vee B))$	replace Q by C in 25
27	$(D \vee (C \vee B)) \to (C \vee (D \vee B))$	replace P by D in 26
28	$(D \vee (C \vee Q)) \to (C \vee (D \vee Q))$	replace B by Q in 27
29	$(D \vee (A \vee Q)) \to (A \vee (D \vee Q))$	replace C by A in 28
30	$(P \vee (A \vee Q)) \to (A \vee (P \vee Q))$	replace D by P in 29
31	$(D \to C) \to (((P \vee (Q \vee A)) \to D) \to ((P \vee (Q \vee A)) \to C))$	replace B by $P \vee (Q \vee A)$ in 21
32	$(D \to (A \vee (P \vee Q))) \to (((P \vee (Q \vee A)) \to D) \to ((P \vee (Q \vee A)) \to (A \vee (P \vee Q))))$	replace C by $A \vee (P \vee Q)$ in 31
33	$((P \vee (A \vee Q)) \to (A \vee (P \vee Q))) \to (((P \vee (Q \vee A)) \to (P \vee (A \vee Q))) \to ((P \vee (Q \vee A)) \to (A \vee (P \vee Q))))$	replace D by $P \vee (A \vee Q)$ in 32
34	$((P \vee (Q \vee A)) \to (P \vee (A \vee Q))) \to ((P \vee (Q \vee A)) \to (A \vee (P \vee Q)))$	MP with 30, 33
35	$(P \vee (Q \vee A)) \to (A \vee (P \vee Q))$	MP with 13, 34
36	$(C \vee (P \vee Q)) \to ((P \vee Q) \vee C)$	replace B by $P \vee Q$ in 3
37	$(A \vee (P \vee Q)) \to ((P \vee Q) \vee A)$	replace C by A in 36
38	$(D \to C) \to ((\neg(P \vee (Q \vee A)) \vee D) \to (\neg(P \vee (Q \vee A)) \vee C))$	replace B by $\neg(P \vee (Q \vee A))$ in 9
39	$(D \to ((P \vee Q) \vee A)) \to ((\neg(P \vee (Q \vee A)) \vee D) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)))$	replace C by $(P \vee Q) \vee A$ in 38
40	$((A \vee (P \vee Q)) \to ((P \vee Q) \vee A)) \to ((\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)))$	replace D by $A \vee (P \vee Q)$ in 39
41	$(\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))$	MP with 37, 40
42	$(C \to (A \vee (P \vee Q))) \to (\neg C \vee (A \vee (P \vee Q)))$	replace B by $A \vee (P \vee Q)$ in 17
43	$((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q)))$	replace C by $P \vee (Q \vee A)$ in 42
44	$(D \to C) \to ((((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to D) \to (((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to C))$	replace B by $(P \vee (Q \vee A)) \to (A \vee (P \vee Q))$ in 21
45	$(D \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))) \to ((((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to D) \to (((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))))$	replace C by $\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)$ in 44
46	$((\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))) \to ((((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q)))) \to (((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))))$	replace D by $\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q))$ in 45
47	$(((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee (A \vee (P \vee Q)))) \to (((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)))$	MP with 41, 46
48	$((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))$	MP with 43, 47
49	$(\neg C \vee ((P \vee Q) \vee A)) \to (C \to ((P \vee Q) \vee A))$	replace B by $(P \vee Q) \vee A$ in 23
50	$(\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)) \to ((P \vee (Q \vee A)) \to ((P \vee Q) \vee A))$	replace C by $P \vee (Q \vee A)$ in 49

51	$(D \to ((P \vee (Q \vee A)) \to ((P \vee Q) \vee A))) \to ((((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to D) \to (((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to ((P \vee (Q \vee A)) \to ((P \vee Q) \vee A))))$	replace C by $(P \vee (Q \vee A)) \to ((P \vee Q) \vee A)$ in 44
52	$((\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)) \to ((P \vee (Q \vee A)) \to ((P \vee Q) \vee A))) \to ((((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))) \to (((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to ((P \vee (Q \vee A)) \to ((P \vee Q) \vee A))))$	replace D by $\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A)$ in 51
53	$(((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to (\neg(P \vee (Q \vee A)) \vee ((P \vee Q) \vee A))) \to (((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to ((P \vee (Q \vee A)) \to ((P \vee Q) \vee A)))$	MP with 50, 52
54	$((P \vee (Q \vee A)) \to (A \vee (P \vee Q))) \to ((P \vee (Q \vee A)) \to ((P \vee Q) \vee A))$	MP with 48, 53
55	$(P \vee (Q \vee A)) \to ((P \vee Q) \vee A)$	MP with 35, 54

□

The associative law for the disjunction (second direction):

Theorem 0.21 (hilb15).

$$((P \vee Q) \vee A) \to (P \vee (Q \vee A))$$

Proof.

1	$(P \vee (Q \vee A)) \to ((P \vee Q) \vee A)$	add sentence hilb14
2	$(P \vee (Q \vee B)) \to ((P \vee Q) \vee B)$	replace A by B in 1
3	$(P \vee (C \vee B)) \to ((P \vee C) \vee B)$	replace Q by C in 2
4	$(D \vee (C \vee B)) \to ((D \vee C) \vee B)$	replace P by D in 3
5	$(D \vee (C \vee P)) \to ((D \vee C) \vee P)$	replace B by P in 4
6	$(D \vee (Q \vee P)) \to ((D \vee Q) \vee P)$	replace C by Q in 5
7	$(A \vee (Q \vee P)) \to ((A \vee Q) \vee P)$	replace D by A in 6
8	$(Q \vee P) \to (P \vee Q)$	add sentence hilb10
9	$(B \vee P) \to (P \vee B)$	replace Q by B in 8
10	$(B \vee C) \to (C \vee B)$	replace P by C in 9
11	$((Q \vee P) \vee C) \to (C \vee (Q \vee P))$	replace B by $Q \vee P$ in 10
12	$((Q \vee P) \vee A) \to (A \vee (Q \vee P))$	replace C by A in 11
13	$(P \to Q) \to (\neg P \vee Q)$	add sentence defimpl1
14	$(\neg P \vee Q) \to (P \to Q)$	add sentence defimpl2
15	$(P \to Q) \to (\neg Q \to \neg P)$	add sentence hilb7
16	$(P \to B) \to (\neg B \to \neg P)$	replace Q by B in 15
17	$(C \to B) \to (\neg B \to \neg C)$	replace P by C in 16
18	$(C \to (A \vee (Q \vee P))) \to (\neg(A \vee (Q \vee P)) \to \neg C)$	replace B by $A \vee (Q \vee P)$ in 17
19	$(((Q \vee P) \vee A) \to (A \vee (Q \vee P))) \to (\neg(A \vee (Q \vee P)) \to \neg((Q \vee P) \vee A))$	replace C by $(Q \vee P) \vee A$ in 18
20	$\neg(A \vee (Q \vee P)) \to \neg((Q \vee P) \vee A)$	MP with 12, 19
21	$(P \to Q) \to ((A \vee P) \to (A \vee Q))$	add axiom axiom4
22	$(P \to Q) \to ((B \vee P) \to (B \vee Q))$	replace A by B in 21
23	$(P \to C) \to ((B \vee P) \to (B \vee C))$	replace Q by C in 22
24	$(D \to C) \to ((B \vee D) \to (B \vee C))$	replace P by D in 23
25	$(D \to C) \to ((((A \vee Q) \vee P) \vee D) \to (((A \vee Q) \vee P) \vee C))$	replace B by $(A \vee Q) \vee P$ in 24
26	$(D \to \neg((Q \vee P) \vee A)) \to ((((A \vee Q) \vee P) \vee D) \to (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)))$	replace C by $\neg((Q \vee P) \vee A)$ in 25

27	$(\neg(A \lor (Q \lor P)) \to \neg((Q \lor P) \lor A)) \to ((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (((A \lor Q) \lor P) \lor \neg((Q \lor P) \lor A)))$	replace D by $\neg(A \lor (Q \lor P))$ in 26
28	$(((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (((A \lor Q) \lor P) \lor \neg((Q \lor P) \lor A))$	MP with 20, 27
29	$(P \lor Q) \to (Q \lor P)$	add axiom axiom3
30	$(P \lor B) \to (B \lor P)$	replace Q by B in 29
31	$(C \lor B) \to (B \lor C)$	replace P by C in 30
32	$(C \lor \neg((Q \lor P) \lor A)) \to (\neg((Q \lor P) \lor A) \lor C)$	replace B by $\neg((Q \lor P) \lor A)$ in 31
33	$(((A \lor Q) \lor P) \lor \neg((Q \lor P) \lor A)) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))$	replace C by $(A \lor Q) \lor P$ in 32
34	$(P \to Q) \to ((A \to P) \to (A \to Q))$	add sentence hilb1
35	$(P \to Q) \to ((B \to P) \to (B \to Q))$	replace A by B in 34
36	$(P \to C) \to ((B \to P) \to (B \to C))$	replace Q by C in 35
37	$(D \to C) \to ((B \to D) \to (B \to C))$	replace P by D in 36
38	$(D \to C) \to ((((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to D) \to ((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to C))$	replace B by $((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))$ in 37
39	$(D \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))) \to (((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to D) \to ((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))))$	replace C by $\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P)$ in 38
40	$((((A \lor Q) \lor P) \lor \neg((Q \lor P) \lor A)) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))) \to ((((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (((A \lor Q) \lor P) \lor \neg((Q \lor P) \lor A))) \to ((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))))$	replace D by $((A \lor Q) \lor P) \lor \neg((Q \lor P) \lor A)$ in 39
41	$((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (((A \lor Q) \lor P) \lor \neg((Q \lor P) \lor A))) \to ((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P)))$	MP with 33, 40
42	$(((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))$	MP with 28, 41
43	$(C \lor ((A \lor Q) \lor P)) \to (((A \lor Q) \lor P) \lor C)$	replace B by $(A \lor Q) \lor P$ in 31
44	$(\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to (((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P)))$	replace C by $\neg(A \lor (Q \lor P))$ in 43
45	$(D \to C) \to (((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to D) \to ((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to C))$	replace B by $\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)$ in 37
46	$(D \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))) \to (((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to D) \to ((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))))$	replace C by $\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P)$ in 45
47	$((((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))) \to (((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to (((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P)))) \to ((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))))$	replace D by $((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P))$ in 46
48	$((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to (((A \lor Q) \lor P) \lor \neg(A \lor (Q \lor P)))) \to ((\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P)))$	MP with 42, 47
49	$(\neg(A \lor (Q \lor P)) \lor ((A \lor Q) \lor P)) \to (\neg((Q \lor P) \lor A) \lor ((A \lor Q) \lor P))$	MP with 44, 48
50	$(P \to B) \to (\neg P \lor B)$	replace Q by B in 13
51	$(C \to B) \to (\neg C \lor B)$	replace P by C in 50
52	$(C \to ((A \lor Q) \lor P)) \to (\neg C \lor ((A \lor Q) \lor P))$	replace B by $(A \lor Q) \lor P$ in 51

53 $((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P))$ — replace C by $A \vee (Q \vee P)$ in 52

54 $(D \to C) \to ((((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to D) \to (((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to C))$ — replace B by $(A \vee (Q \vee P)) \to ((A \vee Q) \vee P)$ in 37

55 $(D \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \to ((((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to D) \to (((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$ — replace C by $\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)$ in 54

56 $((\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)) \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \to ((((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P))) \to (((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))))$ — replace D by $\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P)$ in 55

57 $(((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg(A \vee (Q \vee P)) \vee ((A \vee Q) \vee P))) \to (((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)))$ — MP with 49, 56

58 $((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$ — MP with 53, 57

59 $(\neg P \vee B) \to (P \to B)$ — replace Q by B in 14

60 $(\neg C \vee B) \to (C \to B)$ — replace P by C in 59

61 $(\neg C \vee ((A \vee Q) \vee P)) \to (C \to ((A \vee Q) \vee P))$ — replace B by $(A \vee Q) \vee P$ in 60

62 $(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \to (((Q \vee P) \vee A) \to ((A \vee Q) \vee P))$ — replace C by $(Q \vee P) \vee A$ in 61

63 $(D \to (((Q \vee P) \vee A) \to ((A \vee Q) \vee P))) \to ((((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to D) \to (((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (((Q \vee P) \vee A) \to ((A \vee Q) \vee P))))$ — replace C by $((Q \vee P) \vee A) \to ((A \vee Q) \vee P)$ in 54

64 $((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \to (((Q \vee P) \vee A) \to ((A \vee Q) \vee P))) \to ((((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \to (((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (((Q \vee P) \vee A) \to ((A \vee Q) \vee P))))$ — replace D by $\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)$ in 63

65 $(((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \to (((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (((Q \vee P) \vee A) \to ((A \vee Q) \vee P)))$ — MP with 62, 64

66 $((A \vee (Q \vee P)) \to ((A \vee Q) \vee P)) \to (((Q \vee P) \vee A) \to ((A \vee Q) \vee P))$ — MP with 58, 65

67 $((Q \vee P) \vee A) \to ((A \vee Q) \vee P)$ — MP with 7, 66

68 $(D \to C) \to ((A \vee D) \to (A \vee C))$ — replace B by A in 24

69 $(D \to (Q \vee P)) \to ((A \vee D) \to (A \vee (Q \vee P)))$ — replace C by $Q \vee P$ in 68

70 $((P \vee Q) \to (Q \vee P)) \to ((A \vee (P \vee Q)) \to (A \vee (Q \vee P)))$ — replace D by $P \vee Q$ in 69

71 $(A \vee (P \vee Q)) \to (A \vee (Q \vee P))$ — MP with 29, 70

72 $(C \vee (Q \vee P)) \to ((Q \vee P) \vee C)$ — replace B by $Q \vee P$ in 31

73 $(A \vee (Q \vee P)) \to ((Q \vee P) \vee A)$ — replace C by A in 72

74 $(D \to C) \to (((A \vee (P \vee Q)) \to D) \to ((A \vee (P \vee Q)) \to C))$ — replace B by $A \vee (P \vee Q)$ in 37

75 $(D \to ((Q \vee P) \vee A)) \to (((A \vee (P \vee Q)) \to D) \to ((A \vee (P \vee Q)) \to ((Q \vee P) \vee A)))$ — replace C by $(Q \vee P) \vee A$ in 74

76 $((A \vee (Q \vee P)) \to ((Q \vee P) \vee A)) \to (((A \vee (P \vee Q)) \to (A \vee (Q \vee P))) \to ((A \vee (P \vee Q)) \to ((Q \vee P) \vee A)))$ — replace D by $A \vee (Q \vee P)$ in 75

77	$((A \vee (P \vee Q)) \rightarrow (A \vee (Q \vee P))) \rightarrow ((A \vee (P \vee Q)) \rightarrow ((Q \vee P) \vee A))$	MP with 73, 76
78	$(A \vee (P \vee Q)) \rightarrow ((Q \vee P) \vee A)$	MP with 71, 77
79	$(C \vee A) \rightarrow (A \vee C)$	replace B by A in 31
80	$((P \vee Q) \vee A) \rightarrow (A \vee (P \vee Q))$	replace C by $P \vee Q$ in 79
81	$(D \rightarrow C) \rightarrow ((((P \vee Q) \vee A) \rightarrow D) \rightarrow (((P \vee Q) \vee A) \rightarrow C))$	replace B by $(P \vee Q) \vee A$ in 37
82	$(D \rightarrow ((Q \vee P) \vee A)) \rightarrow ((((P \vee Q) \vee A) \rightarrow D) \rightarrow (((P \vee Q) \vee A) \rightarrow ((Q \vee P) \vee A)))$	replace C by $(Q \vee P) \vee A$ in 81
83	$((A \vee (P \vee Q)) \rightarrow ((Q \vee P) \vee A)) \rightarrow ((((P \vee Q) \vee A) \rightarrow (A \vee (P \vee Q))) \rightarrow (((P \vee Q) \vee A) \rightarrow ((Q \vee P) \vee A)))$	replace D by $A \vee (P \vee Q)$ in 82
84	$(((P \vee Q) \vee A) \rightarrow (A \vee (P \vee Q))) \rightarrow (((P \vee Q) \vee A) \rightarrow ((Q \vee P) \vee A))$	MP with 78, 83
85	$((P \vee Q) \vee A) \rightarrow ((Q \vee P) \vee A)$	MP with 80, 84
86	$(C \rightarrow ((Q \vee P) \vee A)) \rightarrow (\neg((Q \vee P) \vee A) \rightarrow \neg C)$	replace B by $(Q \vee P) \vee A$ in 17
87	$(((P \vee Q) \vee A) \rightarrow ((Q \vee P) \vee A)) \rightarrow (\neg((Q \vee P) \vee A) \rightarrow \neg((P \vee Q) \vee A))$	replace C by $(P \vee Q) \vee A$ in 86
88	$\neg((Q \vee P) \vee A) \rightarrow \neg((P \vee Q) \vee A)$	MP with 85, 87
89	$(D \rightarrow \neg((P \vee Q) \vee A)) \rightarrow ((((A \vee Q) \vee P) \vee D) \rightarrow (((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)))$	replace C by $\neg((P \vee Q) \vee A)$ in 25
90	$(\neg((Q \vee P) \vee A) \rightarrow \neg((P \vee Q) \vee A)) \rightarrow ((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)))$	replace D by $\neg((Q \vee P) \vee A)$ in 89
91	$(((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A))$	MP with 88, 90
92	$(C \vee \neg((P \vee Q) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee C)$	replace B by $\neg((P \vee Q) \vee A)$ in 31
93	$(((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	replace C by $(A \vee Q) \vee P$ in 92
94	$(D \rightarrow C) \rightarrow (((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow D) \rightarrow ((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow C))$	replace B by $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)$ in 37
95	$(D \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow (((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow D) \rightarrow ((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	replace C by $\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)$ in 94
96	$((((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow (((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A))) \rightarrow ((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	replace D by $((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A)$ in 95
97	$((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (((A \vee Q) \vee P) \vee \neg((P \vee Q) \vee A))) \rightarrow ((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 93, 96
98	$(((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	MP with 91, 97
99	$(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A))$	replace C by $\neg((Q \vee P) \vee A)$ in 43
100	$(D \rightarrow C) \rightarrow (((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow C))$	replace B by $\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)$ in 37
101	$(D \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow (((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	replace C by $\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)$ in 100

102	$((((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow (((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A))) \rightarrow ((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	replace D by $((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A)$ in 101
103	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (((A \vee Q) \vee P) \vee \neg((Q \vee P) \vee A))) \rightarrow ((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 98, 102
104	$(\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	MP with 99, 103
105	$(((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))$	replace C by $(Q \vee P) \vee A$ in 52
106	$(D \rightarrow C) \rightarrow (((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow C))$	replace B by $((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)$ in 37
107	$(D \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow (((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	replace C by $\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)$ in 106
108	$((\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow (((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow ((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))))$	replace D by $\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P)$ in 107
109	$((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((Q \vee P) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow ((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)))$	MP with 104, 108
110	$(((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	MP with 105, 109
111	$(\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P))$	replace C by $(P \vee Q) \vee A$ in 61
112	$(D \rightarrow (((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P))) \rightarrow (((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P))))$	replace C by $((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)$ in 106
113	$((\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P))) \rightarrow (((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow ((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P))))$	replace D by $\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)$ in 112
114	$((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow ((((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)))$	MP with 111, 113
115	$(((Q \vee P) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P))$	MP with 110, 114
116	$((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)$	MP with 67, 115
117	$(C \vee P) \rightarrow (P \vee C)$	replace B by P in 31
118	$((A \vee Q) \vee P) \rightarrow (P \vee (A \vee Q))$	replace C by $A \vee Q$ in 117
119	$(D \rightarrow C) \rightarrow ((\neg((P \vee Q) \vee A) \vee D) \rightarrow (\neg((P \vee Q) \vee A) \vee C))$	replace B by $\neg((P \vee Q) \vee A)$ in 24
120	$(D \rightarrow (P \vee (A \vee Q))) \rightarrow ((\neg((P \vee Q) \vee A) \vee D) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	replace C by $P \vee (A \vee Q)$ in 119
121	$(((A \vee Q) \vee P) \rightarrow (P \vee (A \vee Q))) \rightarrow ((\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	replace D by $(A \vee Q) \vee P$ in 120

122	$(\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))$	MP with 118, 121
123	$(((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))$	replace C by $(P \vee Q) \vee A$ in 52
124	$(D \rightarrow C) \rightarrow (((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow C))$	replace B by $((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)$ in 37
125	$(D \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))) \rightarrow (((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))))$	replace C by $\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))$ in 124
126	$((\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))) \rightarrow (((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow ((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))))$	replace D by $\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P)$ in 125
127	$((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee ((A \vee Q) \vee P))) \rightarrow ((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))))$	MP with 122, 126
128	$(((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))$	MP with 123, 127
129	$(\neg C \vee (P \vee (A \vee Q))) \rightarrow (C \rightarrow (P \vee (A \vee Q)))$	replace B by $P \vee (A \vee Q)$ in 60
130	$(\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))) \rightarrow (((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q)))$	replace C by $(P \vee Q) \vee A$ in 129
131	$(D \rightarrow (((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q)))) \rightarrow (((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow D) \rightarrow ((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q)))))$	replace C by $((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q))$ in 124
132	$((\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))) \rightarrow (((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q)))) \rightarrow (((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))) \rightarrow ((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q)))))$	replace D by $\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))$ in 131
133	$((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))) \rightarrow ((((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q))))$	MP with 130, 132
134	$(((P \vee Q) \vee A) \rightarrow ((A \vee Q) \vee P)) \rightarrow (((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q)))$	MP with 128, 133
135	$((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q))$	MP with 116, 134
136	$(C \vee Q) \rightarrow (Q \vee C)$	replace B by Q in 31
137	$(A \vee Q) \rightarrow (Q \vee A)$	replace C by A in 136
138	$(D \rightarrow C) \rightarrow ((P \vee D) \rightarrow (P \vee C))$	replace B by P in 24
139	$(D \rightarrow (Q \vee A)) \rightarrow ((P \vee D) \rightarrow (P \vee (Q \vee A)))$	replace C by $Q \vee A$ in 138
140	$((A \vee Q) \rightarrow (Q \vee A)) \rightarrow ((P \vee (A \vee Q)) \rightarrow (P \vee (Q \vee A)))$	replace D by $A \vee Q$ in 139
141	$(P \vee (A \vee Q)) \rightarrow (P \vee (Q \vee A))$	MP with 137, 140
142	$(D \rightarrow (P \vee (Q \vee A))) \rightarrow ((\neg((P \vee Q) \vee A) \vee D) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))))$	replace C by $P \vee (Q \vee A)$ in 119
143	$((P \vee (A \vee Q)) \rightarrow (P \vee (Q \vee A))) \rightarrow ((\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))))$	replace D by $P \vee (A \vee Q)$ in 142
144	$(\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))$	MP with 141, 143
145	$(C \rightarrow (P \vee (A \vee Q))) \rightarrow (\neg C \vee (P \vee (A \vee Q)))$	replace B by $P \vee (A \vee Q)$ in 51
146	$(((P \vee Q) \vee A) \rightarrow (P \vee (A \vee Q))) \rightarrow (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))$	replace C by $(P \vee Q) \vee A$ in 145

147	$(D \to C) \to (((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to D) \to ((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to C))$	replace B by $((P \vee Q) \vee A) \to (P \vee (A \vee Q))$ in 37
148	$(D \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))) \to (((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to D) \to ((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))))$	replace C by $\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))$ in 147
149	$((\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))) \to (((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))) \to ((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))))$	replace D by $\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q))$ in 148
150	$((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (A \vee Q)))) \to ((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))))$	MP with 144, 149
151	$(((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))$	MP with 146, 150
152	$(\neg C \vee (P \vee (Q \vee A))) \to (C \to (P \vee (Q \vee A)))$	replace B by $P \vee (Q \vee A)$ in 60
153	$(\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))) \to (((P \vee Q) \vee A) \to (P \vee (Q \vee A)))$	replace C by $(P \vee Q) \vee A$ in 152
154	$(D \to (((P \vee Q) \vee A) \to (P \vee (Q \vee A)))) \to (((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to D) \to ((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (((P \vee Q) \vee A) \to (P \vee (Q \vee A)))))$	replace C by $((P \vee Q) \vee A) \to (P \vee (Q \vee A))$ in 147
155	$((\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))) \to (((P \vee Q) \vee A) \to (P \vee (Q \vee A)))) \to (((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))) \to ((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (((P \vee Q) \vee A) \to (P \vee (Q \vee A)))))$	replace D by $\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A))$ in 154
156	$((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (\neg((P \vee Q) \vee A) \vee (P \vee (Q \vee A)))) \to ((((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (((P \vee Q) \vee A) \to (P \vee (Q \vee A))))$	MP with 153, 155
157	$(((P \vee Q) \vee A) \to (P \vee (A \vee Q))) \to (((P \vee Q) \vee A) \to (P \vee (Q \vee A)))$	MP with 151, 156
158	$((P \vee Q) \vee A) \to (P \vee (Q \vee A))$	MP with 135, 157

□

Another consequence from hilb13 is the exchange of preconditions:

Theorem 0.22 (hilb16).

$$(P \to (Q \to A)) \to (Q \to (P \to A))$$

Proof.

1	$(P \vee (Q \vee A)) \to (Q \vee (P \vee A))$	add sentence hilb13
2	$(P \vee (Q \vee B)) \to (Q \vee (P \vee B))$	replace A by B in 1
3	$(P \vee (C \vee B)) \to (C \vee (P \vee B))$	replace Q by C in 2
4	$(D \vee (C \vee B)) \to (C \vee (D \vee B))$	replace P by D in 3
5	$(D \vee (C \vee A)) \to (C \vee (D \vee A))$	replace B by A in 4
6	$(D \vee (\neg Q \vee A)) \to (\neg Q \vee (D \vee A))$	replace C by $\neg Q$ in 5
7	$(\neg P \vee (\neg Q \vee A)) \to (\neg Q \vee (\neg P \vee A))$	replace D by $\neg P$ in 6
8	$(P \to (\neg Q \vee A)) \to (\neg Q \vee (\neg P \vee A))$	reverse abbreviation impl in 7 at occurence 1
9	$(P \to (Q \to A)) \to (\neg Q \vee (\neg P \vee A))$	reverse abbreviation impl in 8 at occurence 1

10	$(P \to (Q \to A)) \to (Q \to (\neg P \vee A))$	reverse abbreviation impl in 9 at occurence 1
11	$(P \to (Q \to A)) \to (Q \to (P \to A))$	reverse abbreviation impl in 10 at occurence 1

□

An analogus form for hilb16:

Theorem 0.23 (hilb17).

$$(Q \to (P \to A)) \to (P \to (Q \to A))$$

Proof.

1	$(P \to (Q \to A)) \to (Q \to (P \to A))$	add sentence hilb16
2	$(P \to (Q \to B)) \to (Q \to (P \to B))$	replace A by B in 1
3	$(P \to (C \to B)) \to (C \to (P \to B))$	replace Q by C in 2
4	$(D \to (C \to B)) \to (C \to (D \to B))$	replace P by D in 3
5	$(D \to (C \to A)) \to (C \to (D \to A))$	replace B by A in 4
6	$(D \to (P \to A)) \to (P \to (D \to A))$	replace C by P in 5
7	$(Q \to (P \to A)) \to (P \to (Q \to A))$	replace D by Q in 6

□

1 Cross Reference

This module is used by the following modules:

Name:	prophilbert2
Version:	1.00.00
Rule version:	1.00.00
Orgin:	prophilbert2_1.00.00_1.00.00.qedeq
pdf:	prophilbert2_1.00.00_1.00.00.pdf