Some more theorems of Predicate Calculus

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Abstract

This module includes first proofs of predicate calculus theorems.

Specification

This document has the following specification:

Name:	predtheo2
Version:	1.00.00
Rule version:	1.00.00
Orgin:	http://www.qedeq.org/0_00_53/predtheo2_1.00.00_1.00.00.qedeq

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References

This document uses the results of the following documents:

Name:	predaxiom
Version:	1.00.00
Rule version:	1.00.00
Orgin:	predaxiom_1.00.00_1.00.00.qedeq
pdf:	predaxiom_1.00.00_1.00.00.pdf
Name:	prophilbert3
Version:	1.00.00
Rule version:	1.00.00
Orgin:	prophilbert3_1.00.00_1.00.00.qedeq
pdf:	prophilbert3_1.00.00_1.00.00.pdf

Content

A simple implication:

Theorem 0.1 (predtheo1).

$$\forall x\, R(x) \rightarrow \exists x\, R(x)$$

Proof.

1	$\forall x\, R(x) \rightarrow R(y)$	add axiom axiom5
2	$R(y) \rightarrow \exists x\, R(x)$	add axiom axiom6
3	$(P \rightarrow Q) \rightarrow ((A \rightarrow P) \rightarrow (A \rightarrow Q))$	add sentence hilb1
4	$(B \rightarrow Q) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow Q))$	replace P by B in 3
5	$(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$	replace Q by C in 4
6	$(B \rightarrow C) \rightarrow ((D \rightarrow B) \rightarrow (D \rightarrow C))$	replace A by D in 5
7	$(R(y) \rightarrow C) \rightarrow ((D \rightarrow R(y)) \rightarrow (D \rightarrow C))$	replace B by $R(y)$ in 6
8	$(R(y) \rightarrow \exists x\, R(x)) \rightarrow ((D \rightarrow R(y)) \rightarrow (D \rightarrow \exists x\, R(x)))$	replace C by $\exists x\, R(x)$ in 7
9	$(R(y) \rightarrow \exists x\, R(x)) \rightarrow ((\forall x\, R(x) \rightarrow R(y)) \rightarrow (\forall x\, R(x) \rightarrow \exists x\, R(x)))$	replace D by $\forall x\, R(x)$ in 8
10	$(\forall x\, R(x) \rightarrow R(y)) \rightarrow (\forall x\, R(x) \rightarrow \exists x\, R(x))$	MP with 2, 9
11	$\forall x\, R(x) \rightarrow \exists x\, R(x)$	MP with 1, 10

$\square$

A well known implication:

Theorem 0.2 (predtheo2).

$$\exists x\, R(x) \rightarrow \neg\forall x\, \neg R(x)$$

Proof.

1	$\forall x\, R(x) \rightarrow R(y)$	add axiom axiom5
2	$\forall x\, \neg R(x) \rightarrow \neg R(y)$	replace $R(@S_1)$ by $\neg R(@S_1)$ in 1
3	$\neg\forall x\, \neg R(x) \lor \neg R(y)$	use abbreviation impl in 2 at occurence 1
4	$(P \lor Q) \rightarrow (Q \lor P)$	add axiom axiom3
5	$(\neg\forall x\, \neg R(x) \lor Q) \rightarrow (Q \lor \neg\forall x\, \neg R(x))$	replace P by $\neg\forall x\, \neg R(x)$ in 4
6	$(\neg\forall x\, \neg R(x) \lor \neg R(y)) \rightarrow (\neg R(y) \lor \neg\forall x\, \neg R(x))$	replace Q by $\neg R(y)$ in 5
7	$\neg R(y) \lor \neg\forall x\, \neg R(x)$	MP with 3, 6
8	$R(y) \rightarrow \neg\forall x\, \neg R(x)$	reverse abbreviation impl in 7 at occurence 1
9	$\exists y\, R(y) \rightarrow \neg\forall x\, \neg R(x)$	Particularize by y in 8
10	$\exists x\, R(x) \rightarrow \neg\forall x\, \neg R(x)$	rename y into x in 9 at occurence 1

$\square$

The reverse is also true:

Theorem 0.3 (predtheo3).

$$\neg\forall x\, \neg R(x) \rightarrow \exists x\, R(x)$$

Proof.

1	$R(y) \to \exists x\, R(x)$	add axiom axiom6
2	$(P \to Q) \to (\neg Q \to \neg P)$	add sentence hilb7
3	$(A \to Q) \to (\neg Q \to \neg A)$	replace P by A in 2
4	$(A \to B) \to (\neg B \to \neg A)$	replace Q by B in 3
5	$(R(y) \to B) \to (\neg B \to \neg R(y))$	replace A by $R(y)$ in 4
6	$(R(y) \to \exists x\, R(x)) \to (\neg\exists x\, R(x) \to \neg R(y))$	replace B by $\exists x\, R(x)$ in 5
7	$\neg\exists x\, R(x) \to \neg R(y)$	MP with 1, 6
8	$\neg\exists x\, R(x) \to \forall y\, \neg R(y)$	Generalize by y in 7
9	$(\neg\exists x\, R(x) \to B) \to (\neg B \to \neg\neg\exists x\, R(x))$	replace A by $\neg\exists x\, R(x)$ in 4
10	$(\neg\exists x\, R(x) \to \forall y\, \neg R(y)) \to (\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x))$	replace B by $\forall y\, \neg R(y)$ in 9
11	$\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)$	MP with 8, 10
12	$\neg\neg P \to P$	add sentence hilb6
13	$\neg\neg Q \to Q$	replace P by Q in 12
14	$\neg\neg\exists x\, R(x) \to \exists x\, R(x)$	replace Q by $\exists x\, R(x)$ in 13
15	$(P \to Q) \to (\neg P \lor Q)$	add sentence defimpl1
16	$(\neg P \lor Q) \to (P \to Q)$	add sentence defimpl2
17	$(P \to Q) \to ((A \lor P) \to (A \lor Q))$	add axiom axiom4
18	$(B \to Q) \to ((A \lor B) \to (A \lor Q))$	replace P by B in 17
19	$(B \to C) \to ((A \lor B) \to (A \lor C))$	replace Q by C in 18
20	$(B \to C) \to ((D \lor B) \to (D \lor C))$	replace A by D in 19
21	$(\neg\neg\exists x\, R(x) \to C) \to ((D \lor \neg\neg\exists x\, R(x)) \to (D \lor C))$	replace B by $\neg\neg\exists x\, R(x)$ in 20
22	$(\neg\neg\exists x\, R(x) \to \exists x\, R(x)) \to ((D \lor \neg\neg\exists x\, R(x)) \to (D \lor \exists x\, R(x)))$	replace C by $\exists x\, R(x)$ in 21
23	$(\neg\neg\exists x\, R(x) \to \exists x\, R(x)) \to ((\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x)))$	replace D by $\neg\neg\forall y\, \neg R(y)$ in 22
24	$(\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x))$	MP with 14, 23
25	$(A \to Q) \to (\neg A \lor Q)$	replace P by A in 15
26	$(A \to B) \to (\neg A \lor B)$	replace Q by B in 25
27	$(\neg\forall y\, \neg R(y) \to B) \to (\neg\neg\forall y\, \neg R(y) \lor B)$	replace A by $\neg\forall y\, \neg R(y)$ in 26
28	$(\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x))$	replace B by $\neg\neg\exists x\, R(x)$ in 27
29	$(P \to Q) \to ((A \to P) \to (A \to Q))$	add sentence hilb1
30	$(B \to Q) \to ((A \to B) \to (A \to Q))$	replace P by B in 29
31	$(B \to C) \to ((A \to B) \to (A \to C))$	replace Q by C in 30
32	$(B \to C) \to ((D \to B) \to (D \to C))$	replace A by D in 31
33	$((\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x)) \to C) \to ((D \to (\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x))) \to (D \to C))$	replace B by $\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x)$ in 32
34	$((\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x))) \to ((D \to (\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x))) \to (D \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x))))$	replace C by $\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x)$ in 33
35	$((\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x))) \to (((\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x))) \to ((\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x))))$	replace D by $\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)$ in 34
36	$((\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \neg\neg\exists x\, R(x))) \to ((\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x)))$	MP with 24, 35
37	$(\neg\forall y\, \neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x))$	MP with 28, 36
38	$(\neg A \lor Q) \to (A \to Q)$	replace P by A in 16
39	$(\neg A \lor B) \to (A \to B)$	replace Q by B in 38
40	$(\neg\neg\forall y\, \neg R(y) \lor B) \to (\neg\forall y\, \neg R(y) \to B)$	replace A by $\neg\forall y\, \neg R(y)$ in 39
41	$(\neg\neg\forall y\, \neg R(y) \lor \exists x\, R(x)) \to (\neg\forall y\, \neg R(y) \to \exists x\, R(x))$	replace B by $\exists x\, R(x)$ in 40

42	$((\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x)) \to C) \to ((D \to (\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x))) \to (D \to C))$	replace B by $\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x)$ in 32
43	$((\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x)) \to (\neg\forall y\,\neg R(y) \to \exists x\, R(x))) \to ((D \to (\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x))) \to (D \to (\neg\forall y\,\neg R(y) \to \exists x\, R(x))))$	replace C by $\neg\forall y\,\neg R(y) \to \exists x\, R(x)$ in 42
44	$((\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x)) \to (\neg\forall y\,\neg R(y) \to \exists x\, R(x))) \to (((\neg\forall y\,\neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x))) \to ((\neg\forall y\,\neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\forall y\,\neg R(y) \to \exists x\, R(x))))$	replace D by $\neg\forall y\,\neg R(y) \to \neg\neg\exists x\, R(x)$ in 43
45	$((\neg\forall y\,\neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\neg\forall y\,\neg R(y) \lor \exists x\, R(x))) \to ((\neg\forall y\,\neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\forall y\,\neg R(y) \to \exists x\, R(x)))$	MP with 41, 44
46	$(\neg\forall y\,\neg R(y) \to \neg\neg\exists x\, R(x)) \to (\neg\forall y\,\neg R(y) \to \exists x\, R(x))$	MP with 37, 45
47	$\neg\forall y\,\neg R(y) \to \exists x\, R(x)$	MP with 11, 46
48	$\neg\forall x\,\neg R(x) \to \exists x\, R(x)$	rename y into x in 47 at occurence 1

□

Exchange of universal quantors:

Theorem 0.4 (predtheo4).

$$\forall x\,\forall y\, R(x,y) \to \forall y\,\forall x\, R(x,y)$$

Proof.

1	$\forall x\, R(x) \to R(y)$	add axiom axiom5
2	$\forall x\, R(x) \to R(w)$	rename y into w in 1
3	$\forall r\, R(r) \to R(w)$	rename x into r in 2 at occurence 1
4	$\forall r\, R(r) \to R(u)$	rename w into u in 3
5	$\forall y\, R(y) \to R(u)$	rename r into y in 4 at occurence 1
6	$\forall y\, R(z,y) \to R(z,u)$	replace $R(@S_1)$ by $R(z,@S_1)$ in 5
7	$\forall x\, R(x) \to R(r)$	rename y into r in 1
8	$\forall s\, R(s) \to R(r)$	rename x into s in 7 at occurence 1
9	$\forall s\, R(s) \to R(z)$	rename r into z in 8
10	$\forall v\, R(v) \to R(z)$	rename s into v in 9 at occurence 1
11	$\forall v\,\forall w\, R(v,w) \to \forall w\, R(z,w)$	replace $R(@S_1)$ by $\forall w\, R(@S_1,w)$ in 10
12	$\forall c\,\forall w\, R(c,w) \to \forall w\, R(z,w)$	rename v into c in 11 at occurence 1
13	$\forall c\,\forall d\, R(c,d) \to \forall w\, R(z,w)$	rename w into d in 12 at occurence 1
14	$\forall c\,\forall d\, R(c,d) \to \forall x_{14}\, R(z,x_{14})$	rename w into x_{14} in 13 at occurence 1
15	$\forall x\,\forall d\, R(x,d) \to \forall x_{14}\, R(z,x_{14})$	rename c into x in 14 at occurence 1
16	$\forall x\,\forall y\, R(x,y) \to \forall x_{14}\, R(z,x_{14})$	rename d into y in 15 at occurence 1
17	$\forall x\,\forall y\, R(x,y) \to \forall y\, R(z,y)$	rename x_{14} into y in 16 at occurence 1
18	$(P \to Q) \to ((A \to P) \to (A \to Q))$	add sentence hilb1
19	$(C \to Q) \to ((A \to C) \to (A \to Q))$	replace P by C in 18
20	$(C \to D) \to ((A \to C) \to (A \to D))$	replace Q by D in 19
21	$(C \to D) \to ((P_7 \to C) \to (P_7 \to D))$	replace A by P_7 in 20
22	$(\forall y\, R(z,y) \to D) \to ((P_7 \to \forall y\, R(z,y)) \to (P_7 \to D))$	replace C by $\forall y\, R(z,y)$ in 21
23	$(\forall y\, R(z,y) \to R(z,u)) \to ((P_7 \to \forall y\, R(z,y)) \to (P_7 \to R(z,u)))$	replace D by $R(z,u)$ in 22
24	$(\forall y\, R(z,y) \to R(z,u)) \to ((\forall x\,\forall y\, R(x,y) \to \forall y\, R(z,y)) \to (\forall x\,\forall y\, R(x,y) \to R(z,u)))$	replace P_7 by $\forall x\,\forall y\, R(x,y)$ in 23

25	$(\forall x\,\forall y\,R(x,y) \;\to\; \forall y\,R(z,y)) \;\to\; (\forall x\,\forall y\,R(x,y) \;\to\; R(z,u))$	MP with 6, 24
26	$\forall x\,\forall y\,R(x,y) \;\to\; R(z,u)$	MP with 17, 25
27	$\forall x\,\forall y\,R(x,y) \;\to\; \forall z\,R(z,u)$	Generalize by z in 26
28	$\forall x\,\forall y\,R(x,y) \;\to\; \forall u\,\forall z\,R(z,u)$	Generalize by u in 27
29	$\forall x\,\forall y\,R(x,y) \;\to\; \forall y\,\forall z\,R(z,y)$	rename u into y in 28 at occurence 1
30	$\forall x\,\forall y\,R(x,y) \;\to\; \forall y\,\forall x\,R(x,y)$	rename z into x in 29 at occurence 1

□

Implication of changing sequence of existence and universal quantor:

Theorem 0.5 (predtheo5).

$$\exists x\,\forall y\,R(x,y) \;\to\; \forall y\,\exists x\,R(x,y)$$

Proof.

1	$\forall x\,R(x) \;\to\; R(y)$	add axiom axiom5
2	$\forall x\,R(x) \;\to\; R(w)$	rename y into w in 1
3	$\forall r\,R(r) \;\to\; R(w)$	rename x into r in 2 at occurence 1
4	$\forall r\,R(r) \;\to\; R(u)$	rename w into u in 3
5	$\forall y\,R(y) \;\to\; R(u)$	rename r into y in 4 at occurence 1
6	$\forall y\,R(x,y) \;\to\; R(x,u)$	replace $R(@S_1)$ by $R(x,@S_1)$ in 5
7	$R(y) \;\to\; \exists x\,R(x)$	add axiom axiom6
8	$R(v) \;\to\; \exists x\,R(x)$	rename y into v in 7
9	$R(v) \;\to\; \exists w\,R(w)$	rename x into w in 8 at occurence 1
10	$R(x) \;\to\; \exists w\,R(w)$	rename v into x in 9
11	$R(x) \;\to\; \exists z\,R(z)$	rename w into z in 10 at occurence 1
12	$R(x,u) \;\to\; \exists z\,R(z,u)$	replace $R(@S_1)$ by $R(@S_1,u)$ in 11
13	$(P \;\to\; Q) \;\to\; ((A \;\to\; P) \;\to\; (A \;\to\; Q))$	add sentence hilb1
14	$(C \;\to\; Q) \;\to\; ((A \;\to\; C) \;\to\; (A \;\to\; Q))$	replace P by C in 13
15	$(C \;\to\; D) \;\to\; ((A \;\to\; C) \;\to\; (A \;\to\; D))$	replace Q by D in 14
16	$(C \;\to\; D) \;\to\; ((P_7 \;\to\; C) \;\to\; (P_7 \;\to\; D))$	replace A by P_7 in 15
17	$(R(x,u) \;\to\; D) \;\to\; ((P_7 \;\to\; R(x,u)) \;\to\; (P_7 \;\to\; D))$	replace C by $R(x,u)$ in 16
18	$(R(x,u) \;\to\; \exists z\,R(z,u)) \;\to\; ((P_7 \;\to\; R(x,u)) \;\to\; (P_7 \;\to\; \exists z\,R(z,u)))$	replace D by $\exists z\,R(z,u)$ in 17
19	$(R(x,u) \;\to\; \exists z\,R(z,u)) \;\to\; ((\forall y\,R(x,y) \;\to\; R(x,u)) \;\to\; (\forall y\,R(x,y) \;\to\; \exists z\,R(z,u)))$	replace P_7 by $\forall y\,R(x,y)$ in 18
20	$(\forall y\,R(x,y) \;\to\; R(x,u)) \;\to\; (\forall y\,R(x,y) \;\to\; \exists z\,R(z,u))$	MP with 12, 19
21	$\forall y\,R(x,y) \;\to\; \exists z\,R(z,u)$	MP with 6, 20
22	$\exists x\,\forall y\,R(x,y) \;\to\; \exists z\,R(z,u)$	Particularize by x in 21
23	$\exists x\,\forall y\,R(x,y) \;\to\; \forall u\,\exists z\,R(z,u)$	Generalize by u in 22
24	$\exists x\,\forall y\,R(x,y) \;\to\; \forall c\,\exists z\,R(z,c)$	rename u into c in 23 at occurence 1
25	$\exists x\,\forall y\,R(x,y) \;\to\; \forall c\,\exists d\,R(d,c)$	rename z into d in 24 at occurence 1
26	$\exists x\,\forall y\,R(x,y) \;\to\; \forall y\,\exists d\,R(d,y)$	rename c into y in 25 at occurence 1
27	$\exists x\,\forall y\,R(x,y) \;\to\; \forall y\,\exists x\,R(x,y)$	rename d into x in 26 at occurence 1

□